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NON-LINEAR TRANSVERSE VIBRATIONS OF A SIMPLY SUPPORTED BEAM CARRYING CONCENTRATED MASSES

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An Euler–Bernoulli beam carrying concentrated masses is considered to be a beam–mass system. The beam is simply supported at both ends. The non-linear equations of motion are derived including stretching due to immovable end conditions. The stretching introduces cubic non-linearities into the equations. Forcing and damping terms are also included. Exact solutions for the natural frequencies are given for the linear problem. For the non-linear problem, an approximate solution using a perturbation method is searched. Non-linear terms of the perturbation series appear as corrections to the linear problem. Amplitude and phase modulation equations are obtained. Non-linear free and forced vibrations are investigated in detail. The effect of the positions, magnitudes and number of the masses are investigated.

1. INTRODUCTION

Non-linear vibrations of a beam having concentrated masses are extensively studied. Beam–mass systems are frequently used as design models in engineering. Exact and approximate analyses have been carried out for calculating the natural frequencies of a beam–mass system under simple supported condition [1–9]. One type of non-linearity, which arises when immovable end conditions are used, is due to the stretching of the beam itself. The non-linear beam vibration studied up to 1979, have been reviewed by Nayfeh and Mook [10]. More recent works on this type are found in references [11–18]. For slightly curved beams with stretching, one may refer to references [19, 20]. It is well known that the analysis of forced vibration of the most simple structural model, in a non-linear way, can reveal non-periodical behaviour, so that even chaotic motions reoccur regularly but not exactly: a small perturbation in initial conditions changes the solution for the so-called sensitivity to initial conditions.

In this study, an Euler–Bernoulli beam carrying concentrated masses is considered as a beam–mass system under simply supported end conditions. Exact natural frequencies are calculated for locations, magnitudes and the number of masses. The method of multiple scales, a perturbation technique, is used to solve the non-linear equations approximately. The amplitude and phase modulation equations are determined from the non-linear analysis. Free and forced vibrations with damping are investigated in detail. The effects of mid-plane stretching on the beam vibrations are studied for different control parameters.

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2. EQUATION OF MOTION

For the system show in Figure 1, the Lagrangian can be written as

$$
\begin{aligned}
£ = & 1/2\sum_{m=0}^{n}\int_{x_m}^{x_{m+1}} \rho A\dot{w}_{m+1}^{*2}\,\mathrm{d}x^* + 1/2\sum_{m=1}^{n} M_m \dot{w}_m^{*2}(x_m, t^*) \\
& - 1/2\sum_{m=0}^{n}\int_{x_m}^{x_{m+1}} EIw_{m+1}^{\prime\prime *2}\,\mathrm{d}x^* - 1/2\sum_{m=0}^{n}\int_{x_m}^{x_{m+1}} EA\left(u_{m+1}^{\prime *} + 1/2w_{m+1}^{\prime *2}\right)^2\mathrm{d}x^*, \\
& x_0 = 0, \quad x_{n+1} = L,
\end{aligned} \tag{1}
$$

where L is the length, ρ is the density, A is the cross-sectional area, E is Young's modulus, I is the second moment of area of the cross-section with respect to the neutral axis, n is the number of concentrated masses, w is the transverse displacement, $(\dot{})$ and $(')$ denote differentiations with respect to time t^* and the spatial variable x^* respectively. The terms in equation (1) are the kinetic energies due to transverse motion of the beam and masses, and elastic energies due to bending and stretching of the beam respectively.

Invoking Hamilton's principle,

$$\delta\int_{t_1}^{t_2} £\,\mathrm{d}t^* = 0 \tag{2}$$

and substituting the Lagrangian from equation (1), performing the necessary algebra and eliminating the axial displacements between equations, one finally obtain the following non-linear coupled integro-differential equations of motion:

$$
\begin{aligned}
\rho A\ddot{w}_{m+1}^* + EIw_{m+1}^{iv*} = & \frac{EA}{2L}\left[\sum_{r=0}^{n}\int_{x_r}^{x_{r+1}} w_{r+1}^{\prime *2}\,\mathrm{d}x^*\right] w_{m+1}^{\prime\prime *} \\
& - \mu^*\dot{w}_{m+1}^* + F_{m+1}^*\cos\Omega^* t^*, \quad m = 0, 1, 2, \ldots, n.
\end{aligned} \tag{3}
$$

There are $(n+1)$ equations in equation (3). Note that the viscous damping coefficient μ^*, and external excitation with amplitude F_{m+1}^* and frequency Ω^* are added to the equations. The boundary conditions can be written for this equation as follows:

$$w_1^*(0, t^*) = w_1^{*\prime}(0, t^*) = 0, \qquad w_{n+1}^*(L, t^*) = w_{n+1}^{*\prime\prime}(L, t^*) = 0, \tag{4}$$

$$w_p^*(x_p, t^*) = w_{p+1}^*(x_p, t^*), \qquad w_p^{*\prime}(x_r, t^*) = w_{p+1}^{*\prime}(x_p, t^*), \qquad w_p^{*\prime\prime}(x_r, t^*) = w_p^{*\prime\prime}(x_p, t^*), \tag{5}$$

$$EIw_p^{*\prime\prime\prime}(x_p, t^*) - EIw_{p+1}^{*\prime\prime\prime}(x_p, t^*) - M_p\ddot{w}_p^*(x_p, t^*) = 0, \qquad p = 1, 2, \ldots n. \tag{6}$$

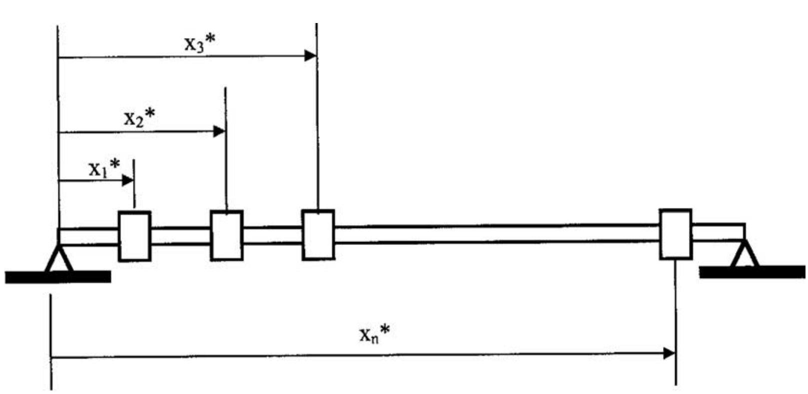

Figure 1. Beam–mass system with simple end conditions.

The equations are made dimensionless using the following definitions:

$$x = x^*/L, \qquad w_p = w_p^*/R, \qquad \eta_p = x_p/L, \tag{7}$$

$$t = \left(1/L^2\right)(EI/\rho A)^{1/2} t^*, \qquad \alpha_p = M_p/\rho AL, \tag{8}$$

where R is the radius of gyration of the beam cross-section with respect to the neural axis and α_p is the ratios of the concentrated masses to the beam mass. Substituting the dimensionless parameters into the equations of motion yield

$$\ddot{w}_{m+1} + w_{m+1}^{iv} = \frac{1}{2}\left[\sum_{r=0}^{n} \int_{\eta_r}^{\eta_{r+1}} w_{r+1}'^2 \, \mathrm{d}x\right] w_{m+1}'' - 2\bar{\mu}\dot{w}_{m+1} + \bar{F}_{m+1} \cos \Omega t, \tag{9}$$

$$w_1(0,t) = w_1''(0,t) = 0, \qquad w_{n+1}(1,t) = w_{n+1}''(1,t) = 0, \tag{10}$$

$$w_p\left(\eta_p, t\right) = w_{p+1}\left(\eta_p, t\right), \qquad w_p'\left(\eta_p, t\right) = w_{p+1}'\left(\eta_p, t\right), \qquad w_p''\left(\eta_p, t\right) = w_{p+1}''\left(\eta_p, t\right), \tag{11}$$

$$w_p'''\left(\eta_p, t\right) - w_{p+1}'''\left(\eta_p, t\right) - \alpha_p \ddot{w}_p\left(\eta_p, t\right) = 0. \tag{12}$$

In equation (9)

$$\eta_0 = 0, \qquad \eta_{n+1} = 1. \tag{13}$$

The solutions and results for different parameters will be presented in the next section.

3. APPROXIMATE ANALYTICAL SOLUTION

In this section, approximate solutions of equations (9)–(13) will be searched with the boundary conditions. The method of multiple scales is applied to partial differential equation systems and boundary conditions directly. Due to the absence of quadratic non-linearities, one assumes expansions of the forms

$$w_{(m+1)}(x,t;\varepsilon) = \varepsilon w_{(m+1)1}(x, T_0, T_2) + \varepsilon^3 w_{(m+1)3}(x, T_0, T_2), \tag{14}$$

where ε is a small book-keeping parameter artificially inserted into the equations. This parameter can be taken to be 1 at the end upon keeping in mind, however, that the deflections are small. Therefore, one investigates a weakly non-linear system. $T_0 = t$ and $T_2 = \varepsilon^2 t$ are the fast and slow time scales. Now consider only the primary resonance case and hence, the forcing and damping terms are ordered so that they counter the effect of the non-linear terms

$$\bar{\mu} = \varepsilon^2 \mu, \qquad \bar{F}_{m+1} = \varepsilon^3 F_{m+1}. \tag{15}$$

The time derivatives are written as

$$(\dot{}) = D_0 + \varepsilon^2 D_2, \qquad (\ddot{}) = D_0^2 + 2\varepsilon D_0 D_2, \qquad D_n = \partial/\partial T_n. \tag{16}$$

Inserting equation (14) into equations (9)–(13) and separating, one obtains

Order (ε):

$$D_0^2 w_{(m+1)1} + w_{(m+1)1}^{iv} = 0, \tag{17}$$

$$w_{11}(0) = w_{11}''(0) = 0, \qquad w_{(n+1)1}(1) = w_{(n+1)1}''(1) = 0, \tag{18}$$

$$w_{p1}(\eta_p) = w_{(p+1)1}(\eta_p), \qquad w_{p1}'(\eta_p) = w_{(p+1)1}'(\eta_p), \qquad w_{p1}''(\eta_p) = w_{(p+1)1}''(\eta_p), \tag{19}$$

$$w_{p1}'''(\eta_p) - w_{(p+1)1}'''(\eta_p) - \alpha_p D_0^2 w_{p1}(\eta_p) = 0. \tag{20}$$

Order (ε^3):

$$D_0^2 w_{(m+1)3} + w_{(m+1)3}^{iv} = -2D_0D_2 w_{(m+1)1} - 2\mu D_0 w_{(m+1)1} + \frac{1}{2}\left[\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} w_{(r+1)1}'^2\,\mathrm{d}x\right] w_{(m+1)1}'' + \bar{F}_{(m+1)}\cos\Omega T_0, \quad (21)$$

$$w_{13}(0) = w_{13}''(0) = 0, \qquad w_{(n+1)3}(1) = w_{(n+1)3}''(1) = 0, \quad (22)$$

$$w_{p3}\left(\eta_p\right) = w_{(p+1)3}\left(\eta_p\right), \qquad w_{p3}'\left(\eta_p\right) = w_{(p+1)3}'\left(\eta_p\right), \qquad w_{p3}''\left(\eta_p\right) = w_{(p+1)3}''\left(\eta_p\right), \quad (23)$$

$$w_{p3}'''\left(\eta_p\right) - w_{(p+1)3}'''\left(\eta_p\right) - \alpha_p D_0^2 w_{p3}\left(\eta_p\right) - 2\alpha_p D_0 D_2 w_{p1}\left(\eta_p\right) = 0. \quad (24)$$

3.1. EXACT SOLUTION TO THE LINEAR PROBLEM

The linear problem is governed by equations (17)–(20). Assuming solutions of the form

$$w_{p1} = \left[A(T_2)\mathrm{e}^{i\omega T_0} + cc\right] Y_p(x), \quad (25)$$

where *cc* stands for complex conjugate and substituting into equations (17)–(20), one obtains

$$Y_{(m+1)}^{iv} - \omega^2 Y_{(m+1)} = 0, \quad (26)$$

$$Y_1(0) = Y_1''(0) = 0, \qquad Y_{(n+1)}(1) = Y_{(n+1)}''(1) = 0, \quad (27)$$

$$Y_p(\eta_p) = Y_{(p+1)}(\eta_p), \qquad Y_p'(\eta_p) = Y_{(p+1)}'(\eta_p), \qquad Y_p''(\eta_p) = Y_{(p+1)}''(\eta_p), \quad (28)$$

$$Y_p'''(\eta_p) - Y_{(p+1)}'''(\eta_p) + \alpha_p\omega^2 Y_p(\eta_p) = 0. \quad (29)$$

Solving equations (26)–(29) exactly for different concentrate masses yield natural frequencies ω. For one masses, the frequency equation is

$$2\tanh\beta\tan\beta + \alpha\beta\{\tanh\beta\sin\beta\eta(\sin\beta\eta - \tan\beta\cos\beta\eta) + \tan\beta\sinh\beta\eta(\tanh\beta\cosh\beta\eta - \sinh\beta\eta)\} = 0. \quad (30)$$

Natural frequency equation for one mass ratio was calculated by Özkaya *et al.* [14]. They obtained the same frequency equation for one mass ratio. For two concentrated masses, the frequency equation is

$$-2\alpha_1\alpha_2\beta^2\cos\beta\cosh\beta + \alpha_1\alpha_2\beta^2\cos(\beta-\eta_1\beta)\cosh\beta + \alpha_1\alpha_2\beta^2\cos(\beta-2\eta_2\beta)\cosh\beta$$
$$+ 2\alpha_1\alpha_2\beta^2\cos[(1-\eta_1-\eta_2)\beta]\cosh[(1-\eta_1-\eta_2)\beta]$$
$$-\alpha_1\alpha_2\beta^2\cos[(1+\eta_1-\eta_2)\beta]\cosh[(1-\eta_1-\eta_2)\beta]$$
$$-2\alpha_1\alpha_2\beta^2\cos[(1-\eta_1-\eta_2)\beta]\cosh[(1+\eta_1-\eta_2)\beta]$$

$$
\begin{aligned}
&+ 2\alpha_1\alpha_2\beta^2 \cos[(1+\eta_1-\eta_2)\beta] \cosh[(1+\eta_1-\eta_2)\beta] \\
&+ \alpha_1\alpha_2\beta^2 \cos\beta \cosh(\beta - 2\eta_1\beta) - \alpha_1\alpha_2\beta^2 \cos(\beta - 2\eta_2\beta) \cosh(\beta - 2\eta_1\beta) \\
&+ \alpha_1\alpha_2\beta^2 \cos\beta \cosh(\beta - 2\eta_2\beta) - \alpha_1\alpha_2\beta^2 \cos(\beta - 2\eta_1\beta) \cosh(\beta - 2\eta_2\beta) \\
&- 4\alpha_1\beta \cosh\beta \sin\beta \\
&- 4\alpha_2\beta \cosh\beta \sin\beta + 4\alpha_1\beta \cosh(\beta - 2\eta_1\beta) \sin\beta + 4\alpha_2\beta \cosh(\beta - 2\eta_2\beta) \sin\beta \\
&- 4\alpha_1\beta \cos\beta \sinh\beta - 4\alpha_2\beta \cos\beta \sinh\beta + 4\alpha_1\beta \cos(\beta - 2\eta_1\beta) \sinh\beta \\
&+ 4\alpha_2\beta \cos(\beta - 2\eta_2\beta) \sinh\beta - 16 \sin\beta \sinh\beta - \alpha_1\alpha_2\beta^2 \sin[(1+2\eta_1-2\eta_2)\beta] \sinh\beta \\
&- \alpha_1\alpha_2\beta^2 \sin(\beta - 2\eta_1\beta) \sinh\beta + \alpha_1\alpha_2\beta^2 \sin(\beta - 2\eta_2\beta) \sinh\beta \\
&+ \alpha_1\alpha_2\beta^2 \sin\beta \sinh[(1+2\eta_1-2\eta_2)\beta] + \alpha_1\alpha_2\beta^2 \sin\beta \sinh(\beta - 2\eta_1\beta) \\
&- \alpha_1\alpha_2\beta^2 \sin\beta \sinh(\beta - 2\eta_2\beta) = 0,
\end{aligned} \tag{31}
$$

where $\beta = \sqrt{\omega}$. The transcendental equation is numerically solved for the first five modes. Results are given in Table 1 for different mass ration. For each case, the natural frequencies are listed for different α_p (the ratio of the concentrated masses to the beam mass) and η_p (the mass location parameter). Natural frequencies were also calculated for three masses (see Table 2). The natural frequencies for various number of frequencies can be calculated using equations (26)–(29).

3.2. NON-LINEAR PROBLEM

Solving order ε^3, one obtains the non-linear corrections to the problem. Because the homogeneous equations (17)–(20) have a non-trivial solution, the non-homogeneous problem (21)–(24) will have a solution only if a solvability condition is satisfied [21]. To determine this condition, first separate the secular and non-secular terms by assuming a

TABLE 1

The first five natural frequencies for two masses with different mass ratios locations

α_1	α_2	η_1	η_2	ω_1	ω_2	ω_3	ω_4	ω_5
1	1	0·1	0·3	6·1182	26·506	55·4118	99·097	196·79
			0·7	6·1834	22·5976	60·226	125·021	174·858
		0·5	0·3	4·7846	19·8023	45·2524	95·2379	158·08
			0·7	4·7297	25·1279	60·8832	141·289	183·11
	10	0·1	0·3	2·5095	26·0754	51·0693	94·5054	194·767
			0·7	2·5165	20·06	58·8238	124·285	168·185
		0·5	0·3	2·4045	13·3671	44·7847	94·7519	158·08
			0·7	2·3875	17·9251	59·5695	136·993	180·905
10	1	0·1	0·3	4·5140	18·5627	38·578	96·6938	195·72
			0·7	4·6714	12·4294	50·9916	121·432	171·647
		0·5	0·3	2·0861	15·9588	43·1699	91·6234	158·043
			0·7	2·0777	22·0363	54·6468	140·866	179·431
	10	0·1	0·3	2·3567	16·2569	29·9752	92·8631	193·92
			0·7	2·41265	8·8503	48·9337	121·018	164·747
		0·5	0·3	1·7707	6·5729	42·9421	94·6427	158·043
			0·7	1·6769	9·8120	53·5165	136·535	177·62

TABLE 2

The first five natural frequencies for three masses with different mass ration and locations

α_1	α_2	α_3	η_1	η_2	η_3	ω_1	ω_2	ω_3	ω_4	ω_5
1	1	1	0·1	0·4	0·8	5·1305	18·915	40·6683	101·949	193·298
1	1	10	0·1	0·4	0·8	3·0114	11·7311	39·4456	98·7132	193·01
1	10	1	0·1	0·4	0·8	2·1818	17·1861	37·3556	99·3226	189·777
10	1	1	0·1	0·4	0·8	4·1416	13·0206	25·9585	99·4389	186·121
10	10	10	0·1	0·4	0·8	1·8639	6·67504	14·1606	93·7742	181·624
1	1	1	0·2	0·5	0·7	4·4113	18·2005	39·1895	137·98	174·375
1	1	10	0·2	0·5	0·7	2·3503	13·4689	35·0008	134·77	171·032
1	10	1	0·2	0·5	0·7	2·0482	18·1854	29·3781	137·958	169·335
10	1	1	0·2	0·5	0·7	2·8578	10·7711	35·3795	137·274	172·9
10	10	10	0·2	0·5	0·7	1·5399	6·3834	13·5785	134·252	164·439

solution of the from

$$w_{(m+1)3} = \phi_{(m+1)}(x, T_2)e^{i\omega T_0} + W_{(m+1)}(x, T_0, T_2) + cc. \tag{32}$$

Substituting this solution into equations (21)–(24), the terms producing secularities are eliminated. Hence the part of the equation determining $\phi_{(m+1)}$ is as follows:

$$\phi^{iv}_{(m+1)} - \omega^2\phi_{(m+1)} = -2\mathrm{i}\omega(A' + \mu A)Y_{(m+1)} + (3/2)A^2\bar{A}\left[\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y'^2_{r+1}\,\mathrm{d}x\right]Y''_p + 1/2F_{(m+1)}e^{i\sigma T_2}, \tag{33}$$

$$\phi_1(0) = \phi''_1(0) = 0, \qquad \phi_{(n+1)}(1) = \phi''_{(n+1)}(1) = 0, \tag{34}$$

$$\phi_p(\eta_p) = \phi_{(p+1)}(\eta_p), \qquad \phi'_p(\eta_p) = \phi'_{(p+1)}(\eta_p), \qquad \phi''_p(\eta_p) = \phi''_{(p+1)}(\eta_p), \tag{35}$$

$$\phi'''_p(\eta_p) - \phi'''_{(p+1)}(\eta_p) + \alpha_p\omega^2\phi_p(\eta_p) - 2\alpha_p\mathrm{i}\omega A' Y_p = 0. \tag{36}$$

In obtaining these equations, one substitutes the first order solutions (25). One also assumed that the external excitation frequency is close to one of the natural frequencies of the system; that is,

$$\Omega = \omega + \varepsilon^2\sigma, \tag{37}$$

where σ is a detuning parameter of order 1. After some algebraic manipulations, one obtains the solvability condition for equations (33)–(36) as

$$2\mathrm{i}\omega(A' + \mu A) + (3/2)b^2A^2\bar{A} + \sum_{r=1}^{n} 2\alpha_r\mathrm{i}\omega A' Y_r^2(\eta_r) - (1/2)fe^{i\sigma T_2} = 0, \tag{38}$$

where the equations are normalized by requiring

$$\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y^2_{r+1}\,\mathrm{d}x = 1. \tag{39}$$

The coefficients b and f in equation (38) are defined as

$$b = \sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} Y'^2_{r+1}\,\mathrm{d}x, \qquad f = \sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}} F_{r+1}Y_{r+1}\,\mathrm{d}x. \tag{40}$$

The complex amplitude A can be written in terms of a real amplitude a and a phase θ

$$A = (1/2)a(T_2)e^{i\theta T_2}. \tag{41}$$

Substituting equation (41) into equation (38), and separating real and imaginary parts, one obtains finally phase and modulation equations

$$\omega a k \gamma' = \omega a \sigma k - \tfrac{3}{16} b^2 a^3 + \frac{1}{2} f \cos\gamma, \qquad \omega a' k = -\omega \mu a + \tfrac{1}{2} f \sin\gamma, \tag{42, 43}$$

where k and γ are defined as

$$k = 1 + \sum_{r=1}^{n} \alpha_r Y_r^2(\eta_r), \qquad \gamma = \sigma T_2 - \theta. \tag{44, 45}$$

In this section amplitude and phase modulation equations are determined from the non-linear analysis for several masses. If one mass has been investigated, same results would be obtained with Özkaya *et al.* [14].

4. NUMERICAL RESULTS

Firstly, the linear natural frequencies for different mass number ($n = 2$ and 3) for various α_p and η_p values are found and given in Table 1. Then, the non-linear frequencies for free undamped vibrations are calculated. In equations (42) and (43), by taking $\mu = f = \sigma = 0$, one obtains

$$a' = 0 \quad \text{and} \quad a = a_0(constant). \tag{46}$$

Note that a_0 is the steady state real amplitude of response. Hence, the non-linear frequency is

$$\omega_{nl} = \omega + \lambda a_0^2, \tag{47}$$

where

$$\lambda = \frac{3}{16}\frac{b^2}{\omega k}. \tag{48}$$

To this order of approximation, thus, the non-linear frequencies have a parabolic relation with the maximum amplitude of vibration. λ can be defined as the non-linear correction coefficient. For different α_p and η_p, the non-linear correction coefficients are listed in Tables 3 and 4 for the first fundamental frequency corresponding to different masses number. λ is a measure of the effect of stretching. The non-linearities are of hardening

TABLE 3

The non-linear frequency correction coefficients for the beam with two masses (first mode)

α_1	α_2	η_1	η_2	λ
1	1	0·1	0·7	1·145060
1	10	0·1	0·7	0·451715
10	1	0·1	0·7	0·836154
10	10	0·1	0·7	0·439536
1	1	0·5	0·3	0·912697
1	10	0·5	0·3	0·441390
10	1	0·5	0·3	0·390698
10	10	0·5	0·3	0·337414

TABLE 4

The non-linear frequency correction coefficients for the beam with three masses (first mode)

α_1	α_2	α_3	η_1	η_2	η_3	λ
1	1	1	0·1	0·4	0·8	0·959973
1	1	10	0·1	0·4	0·8	0·535832
1	10	1	0·1	0·4	0·8	0·416109
10	1	1	0·1	0·4	0·8	0·759183
10	10	10	0·1	0·4	0·8	0·348182
1	1	1	0·2	0·5	0·7	0·826093
1	1	10	0·2	0·5	0·7	0·427772
1	10	1	0·2	0·5	0·7	0·381032
10	1	1	0·2	0·5	0·7	0·514870
10	10	10	0·2	0·5	0·7	0·288347

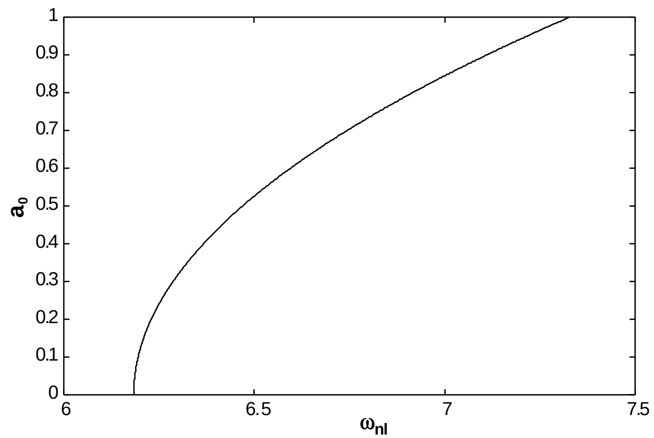

Figure 2. Non-linear frequency versus amplitude for two masses (first mode, $\alpha_1 = 1$, $\alpha_2 = 1$, $\eta_1 = 0{\cdot}1$, $\eta_2 = 0{\cdot}7$).

type. The effect of stretching decreases as α_p increase for all masses number. In Figure 2–5, the non-linear frequency–amplitude curves are drawn for different mass numbers, locations and ratios. In Figures 2 and 3, the non-linear frequency–amplitude curves are drawn for two concentrated mass numbers and different mass ratios. In Figures 4 and 5, non-linear frequency–amplitude curves are drawn for three concentrated masses and different mass ratios. As seen, an increase in concentrated masses decreases the linear and non-linear frequencies.

One can now consider damping and external excitation case. In equations (42) and (43), when the system reaches the steady state region, $a\prime$ and $\gamma\prime$ vanish and hence one obtains

$$\sigma = \frac{3}{16}\frac{a^2b^2}{\omega k} \mp \sqrt{\frac{\tilde{f}^2}{4\omega^2 a^2} - \tilde{\mu}}, \tag{49}$$

where

$$\tilde{f} = f/k, \qquad \tilde{\mu} = \mu/k. \tag{50}$$

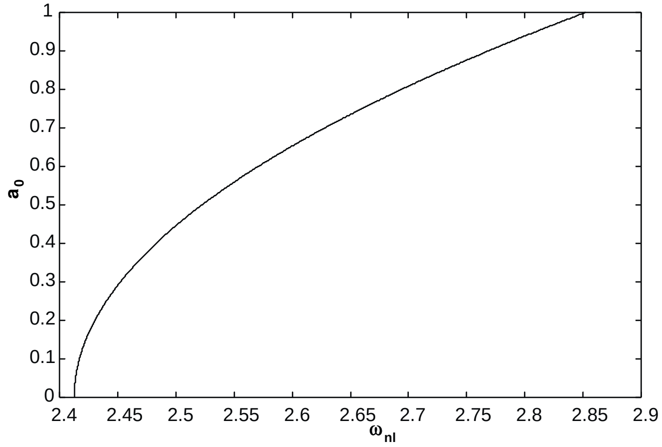

Figure 3. Non-linear frequency versus amplitude for two masses (first mode, $\alpha_1 = 10$, $\alpha_2 = 10$, $\eta_1 = 0{\cdot}1$, $\eta_2 = 0{\cdot}7$).

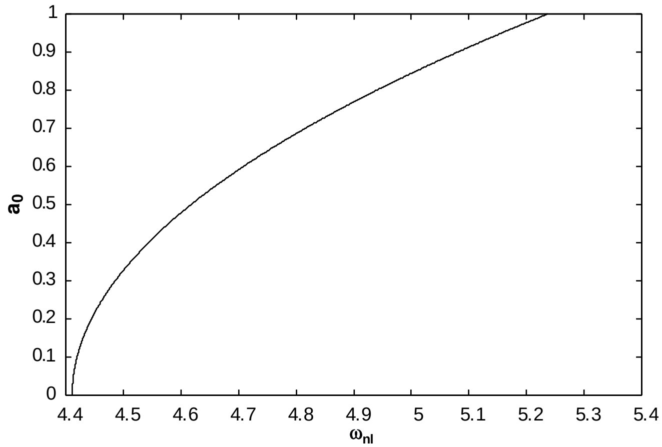

Figure 4. Non-linear frequency versus amplitude for three masses (first mode, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$ $\eta_1 = 0{\cdot}2$, $\eta_2 = 0{\cdot}5$, $\eta_3 = 0{\cdot}7$).

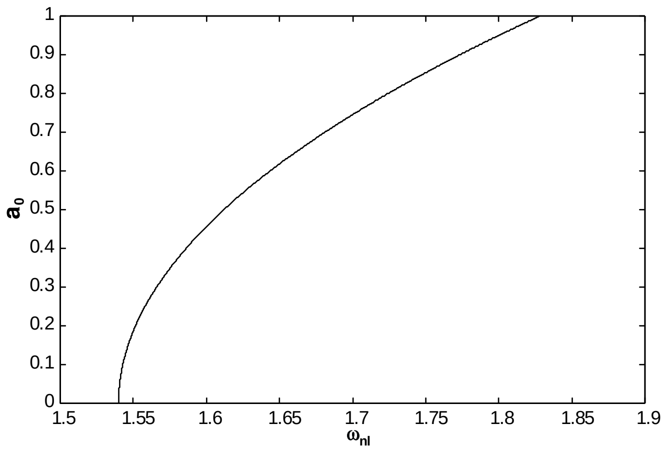

Figure 5. Non-linear frequency versus amplitude for three masses (first mode, $\alpha_1 = 10$, $\alpha_2 = 10$, $\alpha_3 = 10$, $\eta_1 = 0{\cdot}2$, $\eta_2 = 0{\cdot}5$, $\eta_2 = 0{\cdot}7$).

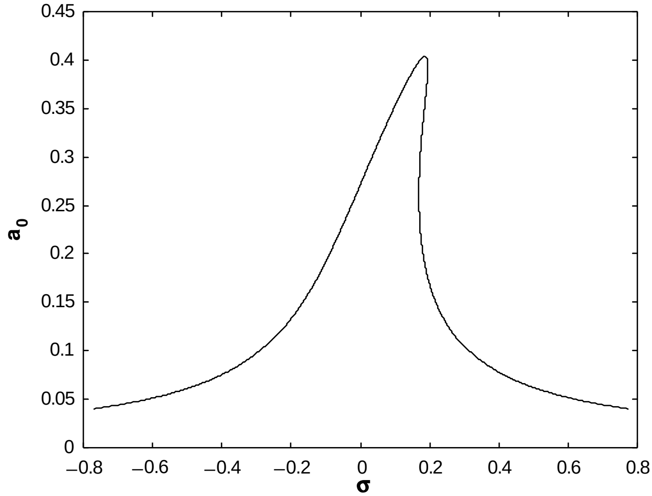

Figure 6. Frequency–response curves for two masses (first mode, $\alpha_1 = 1$, $\alpha_2 = 1$, $\eta_1 = 0{\cdot}1$, $\eta_2 = 0{\cdot}7$).

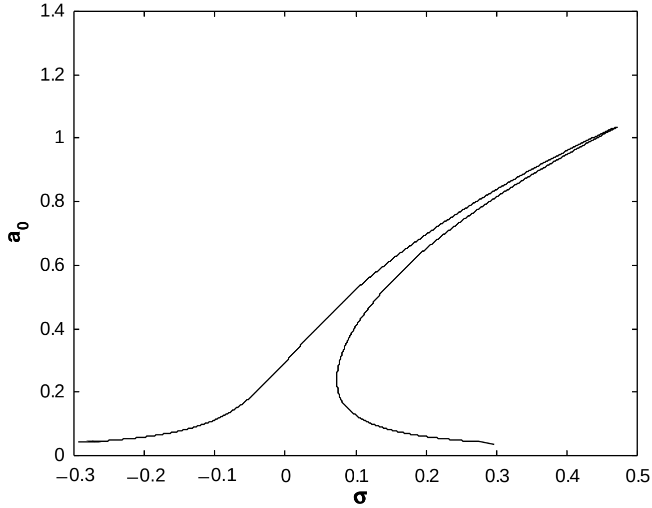

Figure 7. Frequency–response curves for two masses (first mode, $\alpha_1 = 10$, $\alpha_2 = 10$, $\eta_1 = 0{\cdot}1$, $\eta_2 = 0{\cdot}7$).

The detuning parameter shows the nearness of the external excitation frequency to the natural frequency of system. Several figures can be drawn using equation (49). The frequency–response curves are presented for two masses with ratios in Figures 6 and 7 and for three masses with different ratios in Figures 8 and 9. As seen, an increase in concentrated masses increase the maximum amplitude of vibration.

5. CONCLUDING REMARKS

The transverse vibrations of an Euler–Bernoulli beam carrying concentrated masses are investigated. The beam is supported at both ends. The non-linear equations of motion

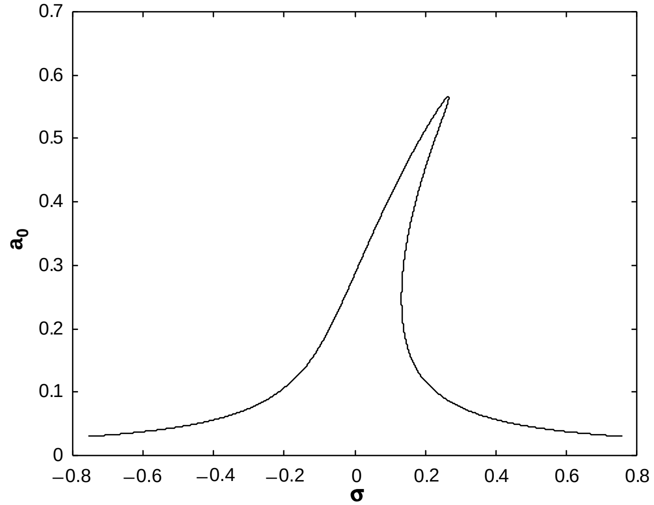

Figure 8. Frequency–response curves for three masses (first mode, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\eta_1 = 0{\cdot}2$, $\eta_2 = 0{\cdot}5$, $\eta_2 = 0{\cdot}7$).

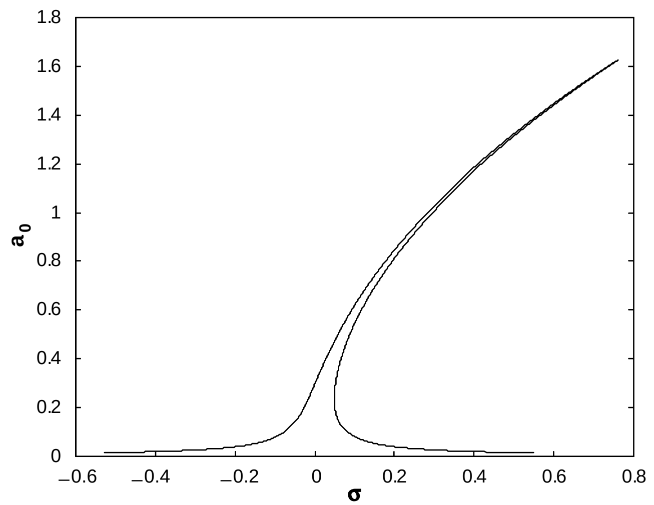

Figure 9. Frequency–response curves for three masses (first mode, $\alpha_1 = 10$, $\alpha_2 = 10$, $\alpha_3 = 10$, $\eta_1 = 0{\cdot}2$, $\eta_2 = 0{\cdot}5$, $\eta_2 = 0{\cdot}7$).

including stretching due to immovable end conditions are derived. Forcing and damping terms are added into the equations. Exact solutions for the natural frequencies are given for the linear part of the problem. For the non-linear problem, approximate solutions using perturbations are searched. Non-linear terms of the perturbation series appear as corrections to linear problem. Non-linear free and forced vibrations are investigated in detail. The effect of the positions, magnitudes and number of the masses are determined.

As the mass ration is increased, the natural and non-linear frequencies decrease. One can observe that the stretching causes a non-linearity of the hardening type. When the ratio mass is increased, the effect of stretching on the non-linear frequencies decreases for all mass numbers. For forced and damped vibrations, since the non-linearity is of hardening type, the frequency–response curves are bent to the right, causing an increase in the multi–valued regions. When the mass ratio and mass number are increased, the multi-valued regions and maximum amplitude increase.

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